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L_p Convergence of Monotone Functions and their Uniform Convergence

JAMES T. LEWIS AND OVED SHISHA

Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881

1. INTRODUCTORY EXAMPLES: A SPECIAL CASE OF THE CONVERGENCE THEOREM

It is well-known that L_p convergence ($0 < p < \infty$) of a sequence of functions $\{f_n\}_{n=1}^{\infty}$ to a continuous limit f does not imply uniform convergence of $\{f_n\}$ to f. This is illustrated by the following simple example.

EXAMPLE 1. For $n = 1, 2,...$ let

$$\begin{aligned} f_n(x) &= 1 + 2n(x - (1/2)), \quad &&\text{if } (1/2) - 1/(2n) \leqslant x < 1/2, \\ &= 1 - 2n(x - (1/2)), \quad &&\text{if } 1/2 \leqslant x < (1/2) + 1/(2n), \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Then for $0 < p < \infty$, $\lim_{n\to\infty} \int_0^1 |f_n(x)|^p \, dx = 0$. However $\{f_n\}$ does not converge uniformly to $f(x) \equiv 0$ on $[0, 1]$ since $f_n(1/2) = 1$ for all $n \geqslant 1$.

One might conjecture that if each f_n is increasing ($f_n(x_1) \leqslant f_n(x_2)$ for $x_1 \leqslant x_2$) then L_p convergence, say in $[0, 1]$, would imply uniform convergence there. A simple example shows this to be false.

EXAMPLE 2. Let $0 < p < \infty$ and, for $n = 1, 2,...,$ $0 \leqslant x \leqslant 1$, let $f_n(x) = x^n$. Again $\lim_{n\to\infty} \int_0^1 |f_n(x)|^p \, dx = 0$ but $\{f_n\}$ does not converge uniformly to $f(x) \equiv 0$ on $[0, 1]$.

In Example 2 the sequence $\{f_n\}$ converges uniformly to f on every closed subinterval of $[0, 1]$ which does not include the right end point $x = 1$. This suggests the following theorem. (All the following integrals are Lebesgue integrals.)

THEOREM 1. *Let f be a real, continuous function on the finite interval (a, b). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of increasing functions on (a, b) such that $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^p \, dx = 0$, where $0 < p < \infty$. Then for any c and d with $a < c < d < b$, the sequence $\{f_n\}$ converges uniformly to f on $[c, d]$.*

Proof. By contradiction. Assume the conclusion is false. Then there exist c, d with $a < c < d < b$, $\epsilon > 0$, a subsequence of $\{f_n\}$ (again denoted by $\{f_n\}$ for convenience) and a sequence $\{x_n\}_{n=1}^{\infty}$ of points in $[c, d]$ such that $|f_n(x_n) - f(x_n)| \geqslant \epsilon$ for all $n \geqslant 1$. The sequence $\{x_n\}$ has a convergent subsequence (again denoted by $\{x_n\}$) with limit, call it y, in $[c, d]$. By the continuity of f at y there exists δ with $0 < \delta \leqslant \min\{c - a, b - d\}$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/3$. There exists an $N \geqslant 1$ such that if $n \geqslant N$ then $|x_n - y| < \delta/2$. Let $n \geqslant N$.

Case 1. $f_n(x_n) \geqslant f(x_n) + \epsilon$. Then for every x in $(y + (\delta/2), y + \delta)$ we have

$$\begin{aligned} f_n(x) &\geqslant f_n(x_n) \geqslant f(x_n) + \epsilon \\ &= (f(x_n) - f(y)) + (f(y) - f(x)) + f(x) + \epsilon \\ &> -(\epsilon/3) - (\epsilon/3) + f(x) + \epsilon = f(x) + (\epsilon/3). \end{aligned}$$

So $f_n(x) - f(x) > \epsilon/3$ for all x in $(y + (\delta/2), y + \delta)$.

Case 2. $f_n(x_n) \leqslant f(x_n) - \epsilon$. By proceeding as in Case 1 one shows that

$$f_n(x) - f(x) < -\epsilon/3 \qquad \text{for all} \quad x \text{ in } (y - \delta, y - (\delta/2)).$$

In either case $|f_n(x) - f(x)| > \epsilon/3$ for all x in an interval of length $\delta/2$. Thus $\int_a^b |f_n(x) - f(x)|^p \, dx > (\epsilon/3)^p \cdot \delta/2$ for all $n \geqslant N$, which contradicts $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^p \, dx = 0$. This completes the proof.

2. The General Convergence Theorem

It is natural to try to generalize Theorem 1 by replacing the monotonicity condition with convexity or with monotonicity of higher order. (f_n is convex on (a, b) iff

$$f_n(\alpha x_1 + (1 - \alpha) x_2) \leqslant \alpha f_n(x_1) + (1 - \alpha) f_n(x_2)$$

whenever $a < x_1 \leqslant x_2 < b$ and $0 \leqslant \alpha \leqslant 1$.) Monotonicity of higher order can be defined in terms of divided differences (a discussion of divided differences can be found in books on numerical analysis, e.g. [1]). The first-order divided difference of f_n is $f_n[x_0, x_1] = [f_n(x_1) - f_n(x_0)]/(x_1 - x_0)$. Clearly f_n is increasing on (a, b) iff $f_n[x_0, x_1] \geqslant 0$ for all distinct x_0, x_1 in (a, b). The second-order divided difference of f_n is

$$f_n[x_0, x_1, x_2] = (f_n[x_1, x_2] - f_n[x_0, x_1])/(x_2 - x_0).$$

It is straightforward to verify that f_n is convex on (a, b) iff $f_n[x_0, x_1, x_2] \geqslant 0$

for all distinct x_0, x_1, x_2 in (a, b). The kth order ($k \geqslant 1$) divided difference of f_n can be expressed by the following formula:

$$f_n[x_0, x_1, ..., x_k] = \sum_{j=0}^{k} \left[f_n(x_j) \Big/ \prod_{\substack{i=0 \\ i \neq j}}^{k} (x_j - x_i) \right]. \qquad (1)$$

Monotonicity of f_n of order $k \geqslant 1$ on (a, b) is the condition that either $f_n[x_0, ..., x_k] \geqslant 0$ for all distinct $x_0, ..., x_k$ in (a, b) or $f_n[x_0, ..., x_k] \leqslant 0$ for all such $x_0, ..., x_k$. (It is known that monotonicity of f_n of order $k \geqslant 2$ on (a, b) implies that $f_n^{(k-2)}$ exists and is continuous on (a, b), cf. [2, p. 381]). Our main result is the generalization of Theorem 1 by replacing "increasing" with monotonicity of order k. For completeness we have included in the statement of the theorem two other conditions which imply monotonicity of order k.

Denote by $\Delta_h{}^k f_n(x)$ the forward difference of kth order of f_n using $x, x + h, ..., x + kh$ (cf. [1, p. 214]).

THEOREM 2. *Let f be a real, continuous function on the finite interval (a, b). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real functions such that $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^p \, dx = 0$, where $0 < p < \infty$. Then for any c and d with $a < c < d < b$, each of the following conditions implies that $\{f_n\}$ converges uniformly to f on $[c, d]$:*

(1) *There exists a positive integer k such that $f_n^{(k)}(x)$ exists and is $\geqslant 0$ for all x in (a, b) and for all $n \geqslant 1$.*

(2) *Each f_n is bounded in (a, b) and there exists a positive integer k such that if $h > 0$ and x and $x + kh$ are in (a, b), then $\Delta_h{}^k f_n(x) \geqslant 0$ for all $n \geqslant 1$.*

(3) *There exists a positive integer k such that $f_n[x_0, ..., x_k] \geqslant 0$ for all distinct $x_0, ..., x_k$ in (a, b) and for all $n \geqslant 1$.*

Proof. We will show that (1) implies (3), (2) implies (3), and (3) implies the conclusion of the theorem.

(a) That (1) implies (3) follows from the fact that if $x_0, ..., x_k$ are distinct points of (a, b) and $n \geqslant 1$, then $f_n[x_0, ..., x_k] = f_n^{(k)}(\xi)/k!$ for some ξ in (a, b) (cf. [1, p. 210]).

(b) That (2) implies (3) is stated in [3, p. 49].

(c) To show that (3) implies the conclusion of the theorem, assume the conclusion is false. Then there exist c and d satisfying $a < c < d < b$, $\epsilon > 0$, a subsequence of $\{f_n\}$ (again denoted by $\{f_n\}$), and a sequence $\{x_n\}_{n=1}^{\infty}$ of points in $[c, d]$ such that $|f_n(x_n) - f(x_n)| \geqslant \epsilon$ for all n. The sequence $\{x_n\}$ has a convergent subsequence (again denoted by $\{x_n\}$) with limit, call it y, in $[c, d]$. By the continuity of f at y there exists δ with $0 < \delta \leqslant \min\{c - a, b - d\}$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/(4k + 2)^{k+1}$.

Let $y_i = y + [i\delta/(2k+1)]$ for all integral i, $-(2k+1) \leqslant i \leqslant 3$. There exists an $N_1 \geqslant 1$ such that for all $n \geqslant N_1$ there is a point $t_{n,i}$ in (y_{2i-1}, y_{2i}), for $i = -k,\ldots,-1$, and a point $t_{n,1}$ in (y_2, y_3) satisfying $|f_n(t_{n,i}) - f(t_{n,i})| < \epsilon/(4k+2)^{k+1}$ for all i. (If not, we would have a subsequence $\{f_{n_j}\}$ with $|f_{n_j}(x) - f(x)| \geqslant \epsilon/(4k+2)^{k+1}$ for all x in an interval of length $\delta/(2k+1)$; this would contradict $\lim_{n\to\infty} \int_a^b |f_n(x) - f(x)|^p \, dx = 0$.) Notice

$$|f_n(t_{n,i}) - f(y)| \leqslant |f_n(t_{n,i}) - f(t_{n,i})| + |f(t_{n,i}) - f(y)| < \frac{2\epsilon}{(4k+2)^{k+1}}.$$

Since $\{x_n\}$ converges to y, there exists an $N_2 \geqslant 1$ such that if $n \geqslant N_2$ then $|x_n - y| < \delta/(2k+1)$. Let $n \geqslant \max\{N_1, N_2\}$.

Case 1. $f_n(x_n) \leqslant f(x_n) - \epsilon$. We show $f_n[t_{n,-k},\ldots,t_{n,-1},x_n] < 0$. For notational convenience set $t_{n,0} \equiv x_n$.

Since a constant can be subtracted from a function without changing its divided difference of any order, we have

$$\begin{aligned}
&f_n[t_{n,-k},\ldots,t_{n,-1},x_n] \\
&\quad = (f_n - f(y))[t_{n,-k},\ldots,t_{n,-1},x_n] \\
&\quad = \frac{f_n(x_n) - f(y)}{\prod_{i=-k}^{-1}(x_n - t_{n,i})} + \sum_{j=-k}^{-1} \frac{f_n(t_{n,j}) - f(y)}{\prod_{i=-k,i\neq j}^{0}(t_{n,j} - t_{n,i})} \\
&\quad = \frac{f_n(x_n) - f(x_n)}{\prod_{i=-k}^{-1}(x_n - t_{n,i})} + \frac{f(x_n) - f(y)}{\prod_{i=-k}^{-1}(x_n - t_{n,i})} + \sum_{j=-k}^{-1} \frac{f_n(t_{n,j}) - f(y)}{\prod_{i=-k,i\neq j}^{0}(t_{n,j} - t_{n,i})} \\
&\quad < \frac{-\epsilon}{(2\delta)^k} + \frac{\epsilon/(4k+2)^{k+1}}{(\delta/(2k+1))^k} + k\,\frac{2\epsilon/(4k+2)^{k+1}}{(\delta/(2k+1))^k} \\
&\quad = \frac{\epsilon}{(2\delta)^k}\left[-1 + \frac{(2k+1)^k\,2^k(1+2k)}{(4k+2)^{k+1}}\right] < 0.
\end{aligned}$$

This contradicts hypothesis (3).

Case 2. $f_n(x_n) \geqslant f(x_n) + \epsilon$. By proceeding as in Case 1 one shows that $f_n[t_{n,-(k-1)},\ldots,t_{n,-1},t_{n,1},x_n] < 0$, a contradiction. This completes the proof.

References

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3. T. Popoviciu, "Les Fonctions Convexes," Hermann, Paris, 1945.